Reification as the Birth of Metaphor*

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> "Basically, I'm not interested in doing research and I never have been. I'm interested in *understanding*, which is quite a different thing." — David Blackwell, referring to his work as a mathematician [Albers & Alexanderson, 1985, p. 19]

1. Introduction: the elusive experience of understanding

The experience of understanding is doubly elusive: it is difficult to achieve and to sustain, and it is even more difficult to capture and to explain. I can clearly remember the event which, for the first time, made me aware of the degree of my ignorance in this respect. I was a beginning teacher and I discovered to my surprise that students who had a good command over systems of linear equations might still be unable to deal with such questions as, "For what value of a parameter q does the given system of linear equations have no solution?". I approached the difficulty casually, confident that we would overcome the problem in an hour or two. My expectations proved wrong. It took days until I felt that the class could cope with parameters. But even then the situation was not what I had hoped for: at the final test only one student managed to produce fully satisfactory solutions to all the problems I posed. In a private conversation with him I remarked: "It seems that you are the only one in this class who really understood the subject." To my distress, the praise was greeted with an angry response: "Wrong! I didn't understand anything. I did what I did but I don't know why it worked". I tried to prove him wrong. I presented him with several other problems, one quite unlike the other, and he solved all of them without a visible difficulty. I claimed that this kind of question just cannot be answered by mechanical application of an algorithm. He kept insisting that he "did not understand anything". We ended up frustrated and puzzled. He felt he did not understand parameters; I sensed that I did not understand understanding.

Reflections on my own experience helped, but only to some extent. I could remember myself as a graduate mathematics student passing exams without difficulty but often feeling that the ease with which I was doing things was not enough to give me the sense of true understanding. Some time later I was happy to find out that even people who grew up to become well-known mathematicians were not altogether unfamiliar with this kind of experience. For example, Paul Halmos [1985] recalls in his "automatography":

> ... I was a student, sometimes pretty good and sometimes less good. Symbols didn't bother ne. I could juggle them quite well ...[but] I was stumped by the infinitesimal subtlety of epsilonic analysis. I could read analytic proofs, remember them if I made an effort, and reproduce them, sort of, but I didn't really know what was going on. [p. 47]

Halmos was fortunate enough to eventually find out what the "real knowing" was all about [Albers & Alexanderson, 1985, p. 123]:

> ... one afternoon something happened. I remember standing at the blackboard in Room 213 of the mathematics building talking with Warren Ambrose and suddenly I understood epsilon. I understood what limits were, and all of that stuff that people were drilling in me became clear. I sat down that afternoon with the calculus textbook by Granville, Smith, and Longley. All of that stuff that previously had not made any sense became obvious ...

Clearly, what people call "true" understanding must involve something that goes beyond the operative ability of solving problems and of proving theorems. But although a person may have no difficulty in diagnosing the degree of his or her understanding, he or she does not find it equally easy to name the criteria according to which such assessment is made. Many works have already been written in which an attempt is made to understand what understanding is all about (for a comprehensive and insightful survey of these see Sierpinska [1993]). In the present paper I will try to take another little step toward capturing the gist of this elusive something that makes us feel we have grasped the essence of a concept, a relation, or a proof.

Let me begin with a few words on the way in which I tackled the question. My quest for a better understanding of mathematical understanding went in two directions. First, I tried to find out what contemporary thinkers have to say on the subject. I soon discovered that, as far as the issue of understanding is concerned, current developments in the psychology of mathematics go hand-in-hand with some of the most significant recent advances in linguistics and in philosophy. The applicability of the latter to the field of mathematical education had already been noted by some researchers [e.g. Dörfler, 1991; Presmeg, 1992]. In this paper I will show how the idea of reification — the basic notion of the conceptual framework on which I have been working for quite a long time now — combines with the new general theories of understanding. I hope to make it clear that the theory of reification is perfectly in tune with the latest philosophical and linguistic developments, and that much may be gained by tightening the links between the different fields. Such a marriage of ideas will be the central theme of the next section.

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